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ANALYTIC STRATIFICATIONS AND THE CUT LOCUS OF A CLASS OF DISTANCE FUNCTIONS

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ABSTRACT

We study the analytic singularities of viscosity solutions of equations of eikonal type and obtain that the analytic singular support of these functions has an analytic stratification. The singular support can be identified with the cut locus of the distance to the boundary of an open set, when the interior is equipped with a degenerate Riemannian metric. We apply the result to elliptic equations as well as to model operators of Grušin type.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with real analytic boundary $\partial\Omega$ and let $A(x)$ be a non-negative $n \times n$ matrix with real analytic entries in $C^\omega(\Omega)$. We consider the solution of the eikonal equation

$$\begin{cases} \langle A(x)\nabla d(x), \nabla d(x)\rangle = 1, & x \in \Omega \\ d(x) = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product and ∇d is the gradient of the function d. Even in the "elliptic" case (i.e. when the matrix $A(x)$ is

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uniformly positive definite), the problem in (1.1) possesses no global differentiable classical solution. Indeed $d \equiv 0$ is not a solution of (1.1) and at an interior minimum (maximum) point for d the gradient of d should be 0, however at such a point equation (1.1) cannot be satisfied in the classical sense.

We adopt the following notion of weak solution introduced by Crandall and Lions in [6].

Definition 1.1: We say that a continuous function $d\colon \overline{\Omega} \to \mathbb{R}$ is a **viscosity subsolution** of (1.1) if for every $\varphi \in C^1(\Omega)$, denoting by $x \in \Omega$ a local maximum point of $d - \varphi$, we have

$$\langle A(x)\nabla\varphi(x), \nabla\varphi(x)\rangle \leq 1.$$

Analogously, d is a **viscosity supersolution** of (1.1) if, for every $\varphi \in C^1(\Omega)$ and x a local minimum point of $d - \varphi$, then

$$\langle A(x)\nabla\varphi(x), \nabla\varphi(x)\rangle \geq 1.$$

Finally, we say that d is a **viscosity solution** if $d = 0$ on $\partial\Omega$ and it is both a viscosity sub- and supersolution.

If $A = \mathrm{Id}$ (the $n \times n$ identity matrix) in (1.1), then the viscosity solution of problem (1.1) is the Euclidean distance function, $d(x)$, from the point $x \in \Omega$ to the boundary $\partial\Omega$. This motivates us to call **distance function** the viscosity solution of equation (1.1). Assume $x \in \Omega \cap \operatorname{sing\,supp}_\omega d$, where $\operatorname{sing\,supp}_\omega d$ denotes the analytic singular support of d (i.e. $x \notin \operatorname{sing\,supp}_\omega d$ if and only if d is real analytic in a suitable neighborhood of x), and let $y \in \partial\Omega$ be such that $d(x) = |x - y|$. Here $|\cdot|$ is the Euclidean norm. One can show that the line segment $\gamma(t) = y + t(x - y)$, $t \geq 0$, is a minimal geodesic from $\partial\Omega$ to x and that such a geodesic ceases to be minimal at the point x. We say that x belongs to the **cut locus** of d in Ω. For a class of matrix functions A, we study the structure of the analytic singular support of d which we call the cut locus of the function d with respect to the possibly degenerate metric $\langle A(x)\xi, \xi\rangle$, or, for the sake of brevity, just the cut locus of d

$$\mathrm{Cut}_A(\Omega) := \{x \in \Omega \mid d \text{ is not real analytic at } x\}. \tag{1.2}$$

In order to describe our results it is useful to introduce the following second order partial differential operator associated to the boundary value problem (1.1)

$$Lu(x) := \mathrm{tr}[A(x)\nabla^2 u(x)] \quad (x \in \Omega), \tag{1.3}$$

where the symbol $\nabla^2 u$ denotes the Hessian matrix of u and "tr" denotes the trace.

Our purpose is to show that the cut locus of the function d in (1.1) is an **analytic stratification**, provided that L is **analytic hypo-elliptic**.

In order to clarify the above statement, we recall that a second order operator L is analytic hypo-elliptic in an open set U if whenever v is a distribution solution of $Lv = f$ in U and $f \in C^\omega(U)$ then v is also in $C^\omega(U)$.

The problem of the analytic hypo-ellipticity of second order differential operators with semi-definite quadratic form and analytic coefficients is one of the major problems in the theory of linear PDE's and is far from being understood. There are though classes of operators for which C^ω-hypo-ellipticity has been proved (see e.g. [19] and [20]).

Finally, loosely speaking, a set admits an analytic stratification if it splits as the disjoint union of a locally finite family of real analytic manifolds (see Definition 2.1 below).

Our approach is somewhat indirect. Our goal is to show that the solution to our problem is actually a subanalytic function. We refer to the paper by Tamm [17] for the definitions and the main properties of subanalytic functions (see also Bierstone and Milman [4]). One of the results proved in [17] is that the singular set of a subanalytic function is a subanalytic set and as such it is an analytic stratification. Thus the problem of showing that the set (1.2) has an analytic stratification is reduced to that of showing that the solution of equation (1.1) is **subanalytic** (see Theorem 2.1 below).

We point out that, in the case $A > 0$, the subanalyticity of d is a consequence of Theorem 3.5.2 of [17]. Sussmann [15] and Agrachev–Gauthier [1] have proved the subanalyticity of the point-to-point distance associated to a class of sub-Riemannian vector fields.

We would also like to mention that the problem of the regularity of the distance to the boundary has been the object of a number of papers, the most recent of which, [13] and [14], deal with problems closely related to ours. More specifically Li and Nirenberg in [13] are concerned with the finiteness of the $(n-1)$-dimensional Hausdorff measure for the $C^{1,1}$ analog of our $\mathrm{Cut}_A(\Omega)$—or *ridge set* in their terminology. We also would like to say that in [13] a Hamilton–Jacobi equation is considered which does not contain degenerate sums of squares of vector fields.

In [14] a 2-dimensional real analytic case is also studied and it is proved that the cut locus is the union of isolated points and linear graphs. The authors give also an example showing that if the boundary of a set is a $C^{1,1}$ convex curve then the analytic singular support of the Euclidean distance to that curve has non-zero Lebesgue measure.

In the present paper, we take a purely PDE approach to the problem of the subanalyticity of the solution of (1.1) for a class of equations including non-sub-Riemannian sums of squares of vector fields. This implies that the vector fields we consider may not span a subbundle of the tangent bundle.

Furthermore, we remark that any lifting of our fields to a sub-Riemannian set might compromise the hypoanalyticity property of the "sum of squares" operator, thus preventing us from using that technique.

Our main idea is to deduce a suitable representation formula for d. Using such a formula, the analytic hypo-ellipticity of L and a general result on subanalytic objects due to Tamm [17], the subanalyticity of the distance function ensues. This is carried out in Section 2.

In Section 3 we show that our theorem applies to the elliptic case, i.e. when $A > 0$. Then in Section 4 we show that our approach also applies to the degenerate case. For the sake of simplicity, we limit our attention to the case when L is the Grušin operator, i.e. $L = \sum_{j=1}^{n}(\partial_{x_j}^2 + x_j^{2k}\partial_y^2)$, $k \geq 1$ integer (see Section 4).

2. The basic result

2.1. Preliminaries. We begin by recalling some basic facts on subanalytic subsets and analytic stratification. We follow Tamm [17] (another account of these topics is given by Bierstone and Milman [4]).

Definition 2.1: By a **stratification** of a set $S \subset \Omega$ we mean a locally finite decomposition

$$S = \bigcup_{j=1}^{\nu} V_j$$

for some $\nu \in \mathbb{N} \cup \{\infty\}$, into a disjoint union of connected real analytic submanifolds V_j (the **strata** of S) such that $\dim V_j \leq n-1$ and

$$\overline{V_j} \cap V_k \neq \emptyset \quad \Longrightarrow \quad V_k \subset \overline{V_j} \quad \text{and} \quad \dim V_k \leq \dim V_j - 1.$$

The next result plays a crucial role in our reduction.

Theorem 2.1: *Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected set with real analytic boundary and let $v\colon \overline{\Omega} \times [0, \varepsilon_*] \to \mathbb{R}$ be a continuous function, real analytic* *on* $\Omega\times]0,\varepsilon_*[$. *Here* ε_* *is a positive number. Set*

$$d(x) := \min_{\varepsilon\in[0,\varepsilon_*]} v(x,\varepsilon) \quad (x \in \overline{\Omega}).$$

Then, the set $C = \{x \in \Omega \mid d \textbf{ is not real analytic at } x\}$ *admits an analytic stratification.*

Proof: Let us denote by $\tilde{v} = v_{|\Omega\times]0,\varepsilon_*[}$ the restriction of v to the open set $\Omega\times]0,\varepsilon_*[$. Since $\tilde{v}$ is analytic and Ω is an analytic manifold the graph of $\tilde{v}$ is a semianalytic set. Now the graph of the function v is the closure in $\mathbb{R}^{n+2}$ of the graph of $\tilde{v}$, and since the closure of a semianalytic set is semianalytic by Corollary 2.8 in [4], we obtain that the function v is semianalytic. In particular v is a subanalytic function on $\overline{\Omega} \times [0,\varepsilon_*]$.

Next we need the following remark of Bierstone and Milman ([4]):

Lemma 2.1 (see [4], Remark 3.11 (2)): *Let M and N be real analytic manifolds and let X and T be subanalytic sets of M and N, respectively, where T is compact. If $f\colon X \times T \to \mathbb{R}$ is a continuous subanalytic function, it follows that*

$$g(x) = \min_{t\in T} f(x,t)$$

is a subanalytic function on X.

Using this fact we obtain that $d(x) = \min_{\varepsilon\in[0,\varepsilon_*]} v(x,\varepsilon)$ is a subanalytic function on $\overline{\Omega}$. Applying the results of [17] we have that $\operatorname{sing\,supp}_\omega d$ is a subanalytic subset of $\overline{\Omega}$ and as such $\operatorname{sing\,supp}_\omega d$ has an analytic stratification by [10]. ∎

2.2. Assumptions. In order to single out the class of operators we deal with, let us consider the following Dirichlet problem

$$\begin{cases} -\varepsilon Lu + \langle A\nabla u, \nabla u\rangle = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

This is a vanishing viscosity approximation of the eikonal equation (1.1). Here,

$$Lu(x) = \operatorname{tr}[A(x)\nabla^2 u(x)].$$

We denote by $u(x,\varepsilon)$ the solution of the above problem depending on the small positive parameter ε.

In the following we establish a general framework by assuming conditions (H1)–(H3) below. We stress that (H1)–(H3) have just the role of putting in evidence what we need our operators to satisfy. Further in this paper we produce classes of differential operators that satisfy (H1)–(H3).

(H1) $A(x)$ is a non-negative $n \times n$ matrix with real analytic entries defined in Ω.

The assumption (H1) can be rephrased saying that we are working with elliptic (possibly degenerate) operators.

(H2) There exists $\varepsilon_* > 0$ such that

(a) For every $\varepsilon \in]0, \varepsilon_*]$, the Dirichlet problem (2.1) has a (classical) solution $u(x, \varepsilon)$ continuous on $\overline{\Omega}$.

(b) for every $a \in]0, \varepsilon_*[$ and for every $x_0 \in \Omega$, we can find a neighborhood V of x_0, $V \subset\subset \Omega$, and a constant $C > 0$ such that for every $(x, \varepsilon) \in V \times]a, \varepsilon_*[$,

$$|\partial_x^\alpha u(x, \varepsilon)| \le C^{|\alpha|+1} |\alpha|! \tag{2.2}$$

for every $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ where, as usual, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$\partial_x^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Remark 2.1: (H2) is an existence (a) and regularity (b) assumption on the solutions of the approximating problems (2.1). More precisely, (H2)(b) is an analytic hypo-ellipticity assumption on the operator L. Indeed, in concrete problems (see Sections 3 and 4) Condition (2.2) is equivalent to stating that L is analytic hypo-elliptic.

(H3) There exist $\alpha, \beta \in]0, 1]$ and $c > 0$ such that

$$[u]_\alpha := \sup_{x \ne y, x, y \in \overline{\Omega}} \frac{|u(x, \varepsilon) - u(y, \varepsilon)|}{|x - y|^\alpha} \le c \quad \forall \varepsilon \in]0, \varepsilon_*], \tag{2.3}$$

$$\|u(\cdot, \varepsilon) - u(\cdot, \varepsilon')\|_{L^\infty} \le c\varepsilon^\beta \quad \forall \varepsilon \in 0 < \varepsilon' \le \varepsilon \le \varepsilon_*. \tag{2.4}$$

Remark 2.2: As shown in Proposition 2.2 below, the above estimates imply the existence of the viscosity solution of equation (1.1).

2.3. Remarks on the assumption (H2). It seems to be pretty hard to check the assumption (H2) in the present form. A useful strategy is to reduce the problem (2.1) to a linear one. Let us consider the following Dirichlet problem

$$\begin{cases} \varepsilon^2 L f = f & \text{in } \Omega, \\ f = 1 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

We denote by $f(x, \varepsilon)$ the solution of the above problem depending on the small positive parameter ε.

PROPOSITION 2.1: *Assume (H1), let Ω be an open bounded set and suppose that there exists a positive number ε_* such that*

(a) *For every $\varepsilon \in]0, \varepsilon_*]$, the Dirichlet problem (2.5) has a (classical) solution $f(x, \varepsilon)$ continuous on $\overline{\Omega}$.*

(b) *For every $a \in]0, \varepsilon_*[$ and for every $x_0 \in \Omega$, we can find a neighborhood V of x_0, $V \subset\subset \Omega$, and a constant $C > 0$ such that for every $(x, \varepsilon) \in V \times]a, \varepsilon_*[$,*

$$|\partial_x^\alpha f(x, \varepsilon)| \leq C^{|\alpha|+1} \alpha!.$$

(c) *There exists a function of class C^1, ψ, such that*

$$\langle A(x)\nabla\psi(x), \nabla\psi(x)\rangle \neq 0 \quad \forall x \in \Omega. \tag{2.6}$$

Then, the assumption (H2) holds.

Proof: Condition (c) above implies that $f > 0$ in $\overline{\Omega}$. In fact, set

$$v(x) = e^{\lambda(\psi(x) - c)}, \quad x \in \overline{\Omega}$$

where ψ is the function in (2.6), c is a suitable constant such that $\psi - c < 0$ in Ω and λ is a positive constant yet to be determined. Let us consider the minimum of the function $f - v$ in $\overline{\Omega}$. It is clear that, if the minimum value of $f - v$, m, is positive then

$$f(x) \geq v(x) + m > 0 \quad \forall x \in \overline{\Omega}$$

and we are done. Assume now that $m \leq 0$. Let x_m be one of the points where this minimum is attained. Since $f - v_{|\partial\Omega} > 0$, then necessarily $x_m \in \Omega$. Now, we have that

$$f(x_m) - v(x_m) = \varepsilon^2 Lf(x_m) - v(x_m),$$

$\nabla f(x_m) = \lambda v(x_m)\nabla\psi(x_m)$ and

$$\nabla^2 f(x_m) \geq \lambda^2 v(x_m)\nabla\psi(x_m) \otimes \nabla\psi(x_m) + \lambda v(x_m)\nabla^2\psi(x_m).$$

Hence,

$$\begin{aligned} f(x_m) - v(x_m) &= \varepsilon^2 Lf(x_m) - v(x_m) \\ &\geq v(x_m)\{\varepsilon^2\lambda^2 c_1 + \varepsilon^2\lambda \operatorname{tr}[A(x_m)\nabla^2\psi(x_m)] - 1\} \end{aligned}$$

where

$$c_1 := \min_{x \in \overline{\Omega}} \langle A(x)\nabla\psi(x), \nabla\psi(x)\rangle > 0.$$

Thus it is clear that, taking λ large enough, we obtain

$$\{\varepsilon^2\lambda^2 c_1 + \varepsilon^2\lambda\,\mathrm{tr}[A(x_m)\nabla^2\psi(x_m)] - 1\} > 0,$$

and hence a contradiction, so that $f > 0$ in $\overline{\Omega}$. It is then straightforward to deduce that

$$u(x,\varepsilon) = -\varepsilon\log f(x,\varepsilon), \quad (x,\varepsilon)\in\overline{\Omega}\times[0,\varepsilon_*] \tag{2.7}$$

and the conclusion easily follows. ∎

2.4. THE ASSUMPTION (H3). We consider the equation

$$\begin{cases} \langle A(x)\nabla d(x), \nabla d(x)\rangle = 1, & x\in\Omega,\\ d(x) = 0, & x\in\partial\Omega. \end{cases} \tag{2.8}$$

The following (existence) result is a straightforward consequence of our assumptions.

PROPOSITION 2.2: *Let $\Omega\subset\mathbb{R}^n$ be an open bounded set and assume that* (H2)(a) *and* (H3) *hold. Then, there exists a unique viscosity solution d of the problem* (2.8) *and*

$$\lim_{\varepsilon\to 0}\sup_{x\in\Omega}|d(x) - u(x,\varepsilon)| = 0.$$

Proof: Because of Assumption (H2)(a) we have the existence of the function $u(x,\varepsilon)$. Moreover, using the assumption (H3), it is easy to see that the family of functions $\{u(\cdot,\varepsilon)\}_\varepsilon$ is uniformly bounded and continuous. Hence, by the Arzelà–Ascoli Theorem, there exists $\lim_{\varepsilon\to 0^+} u(x,\varepsilon)$, for every $x\in\overline{\Omega}$. Finally, the fact that such a limit is the viscosity solution of equation (2.8) is a standard computation. The uniqueness of the solution to the problem (2.8) is a well-known result of the theory of viscosity solutions (it suffices to use the Kružkov transformation and the standard doubling variables argument). ∎

2.5. THE MAIN RESULT. The remaining part of this section is devoted to the proof of the following

THEOREM 2.2: *Under the assumptions* (H1), (H2) *and* (H3) *let $\Omega\subset\mathbb{R}^n$ be an open, bounded set with real analytic boundary. Then, $\mathrm{Cut}_A(\Omega)$ admits an analytic stratification.*

For $\varepsilon\in\,]0,\varepsilon_*]$, we denote by $u(x,\varepsilon)$ the solution of a vanishing viscosity approximation of the eikonal equation (2.8)

$$\begin{cases} -\varepsilon Lu(x) + \langle A(x)\nabla u(x), \nabla u(x)\rangle = 1 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

PROPOSITION 2.3: *Under the assumptions* (H1), (H2) *and* (H3), *the solution* u *of the boundary value problem* (2.9) *is real analytic w.r.t. the variables* $(x, \varepsilon) \in \Omega\times]0, \varepsilon_*[$.

Proof: As a consequence of equation (2.7) and the fact that the composition of real analytic functions is real analytic it suffices to show that the solution of (2.5) is real analytic. Defining

$$\delta = 1/\varepsilon^2 \quad \text{and} \quad \delta_* = 1/\varepsilon_*^2$$

and using once more the fact that the composition of real analytic functions is real analytic, the proof reduces to showing that the solution v of the following problem is real analytic w.r.t. $\delta \in]\delta_*, +\infty[$ (uniformly w.r.t. x)

$$\begin{cases} Lv = \delta v & \text{in } \Omega, \\ v = 1 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

Once more, we emphasize that the solution of (2.10) is non-negative.

It is well known that in order to show that a function is real analytic it suffices to verify that it is a C^∞ function, it is real analytic with respect to each variable separately and satisfies uniform estimates on the derivatives. Now, arguing by induction, let us show that v is of class C^∞ with respect to the variables (x, δ).

First, we show that v is continuous. We remark that, by the assumption (H3), v is uniformly (w.r.t. δ) bounded. We have that at a maximum point x_M of $v(\cdot, \delta) - v(\cdot, \delta')$

$$0 \geq L[v(x_M, \delta) - v(x_M, \delta')] = \delta[v(x_M, \delta) - v(x_M, \delta')] + (\delta - \delta')v(x_M, \delta')$$

while at a minimum point

$$0 \leq L[v(x_m, \delta) - v(x_m, \delta')] = \delta[v(x_m, \delta) - v(x_m, \delta')] + (\delta - \delta')v(x_m, \delta'),$$

i.e.

$$\delta \max_{x\in\overline{\Omega}} |v(x, \delta) - v(x, \delta')] \leq C|\delta - \delta'|$$

($C > 0$ is a constant bigger than the L^∞ norm of v), and the continuity of v follows.

Now, let us show that $v \in C^k \Longrightarrow v \in C^{k+1}$. Suppose that $v \in C^k$; we have that

$$\begin{cases} L\partial_\delta^k v = \delta\partial_\delta^k v + k\partial_\delta^{k-1} v & \text{in } \Omega, \\ \partial_\delta^k v = 0 & \text{on } \partial\Omega. \end{cases}$$

Let g be the solution of the problem

$$\begin{cases} Lg = \delta g + (k+1)\partial_\delta^k v & \text{in } \Omega, \\ g = 0 & \text{on } \partial\Omega. \end{cases}$$

Arguing as for v it is easy to show that g is continuous w.r.t. δ. Now, we want to show that

$$\lim_{h\to 0} \max_{x\in\overline{\Omega}} |\frac{\partial_\delta^k v(x,\delta+h) - \partial_\delta^k v(x,\delta)}{h} - g(x,\delta)| = 0.$$

Set

$$w(x) := \frac{\partial_\delta^k v(x,\delta+h) - \partial_\delta^k v(x,\delta)}{h} - g(x,\delta),$$

and observe that $w = 0$ on $\partial\Omega$ and, in Ω,

$$Lw = \delta w + k\frac{\partial_\delta^{k-1} v(x,\delta+h) - \partial_\delta^{k-1} v(x,\delta)}{h} + \partial_\delta^k v(x,\delta+h) - (k+1)\partial_\delta^k v(x,\delta).$$

Once more computing w at a maximum and minimum point, using the equation (and the induction assumption) we conclude that $v \in C^{k+1}$. Hence, $v \in C^\infty$.

Now, the proof reduces to showing that the function v is real analytic w.r.t. δ uniformly in x. We use the maximum principle and a result of Bernstein on real analytic functions of one variable (see [3]).

Arguing by induction it is easy to see that, for every non-negative integer k,

(2.11) $\quad \partial^k v/\partial\delta^k$ is non-negative for k even and is non-positive for k odd.

Hence, we have a smooth function v with the above sign property of the derivatives and we want to conclude that v is real analytic. We describe the argument in the case of k even, the case of k odd being completely analogous. Take a and b such that $\delta_* < b < a < +\infty$. We want to show that v is real analytic at a. The Taylor's formula yields that, for some $\xi \in]a,b[$,

$$v(b) = \sum_{j=0}^{k-1} \frac{1}{j!}\frac{\partial^j v}{\partial\delta^j}(a)(b-a)^j + \frac{1}{k!}\frac{\partial^k v}{\partial\delta^k}(\xi)(b-a)^k,$$

hence the sign condition (2.11) yields that

$$v(b) \geq \frac{1}{k!}\frac{\partial^k v}{\partial\delta^k}(\xi)(b-a)^k \geq 0$$

and using the fact that the odd derivatives are non-negative (so the k derivative is a decreasing function) we deduce that

$$\left|\frac{\partial^k v}{\partial\delta^k}(a)\right| \leq v(b)k!\Big(\frac{1}{b-a}\Big)^k.$$

That is we get the real analyticity w.r.t. δ (uniformly w.r.t. x). This completes our proof. ∎

Proof of Theorem 2.2: Assumption (H3) (2.4) yields

$$\|u(\cdot,\varepsilon)-u(\cdot,\varepsilon')\|_{L^\infty}\le c\varepsilon^\beta,\quad 0<\varepsilon'\le\varepsilon\le\varepsilon_*,$$

hence taking the limit, as $\varepsilon'\to 0$, in the above inequality we obtain

$$\|u(\cdot,\varepsilon)-d(\cdot)\|_{L^\infty}\le c\varepsilon^\beta\quad(\varepsilon\in]0,\varepsilon_*]).$$

Hence,

$$d(x)=\inf_{\varepsilon\in[0,\varepsilon_*]}\{u(x,\varepsilon)+c\varepsilon^\beta\}\quad(x\in\Omega).$$

Now, in view of Proposition 2.3, $u(x,\varepsilon)+c\varepsilon^\beta$ is real analytic for $(x,\varepsilon)\in\Omega\times]0,\varepsilon_*[$. Moreover, it is a continuous function on $\overline{\Omega}\times[0,\varepsilon_*]$. Hence, the conclusion follows by Theorem 2.1. ∎

3. The elliptic case

In this section we show how our abstract result applies to the elliptic case $A>0$. We assume

(H) $A(x)$ is a $n\times n$ matrix with real analytic entries and, for some $c>0$,

$$\langle A(x)\xi,\xi\rangle\ge c\|\xi\|^2\quad\forall(x,\xi)\in\overline{\Omega}\times\mathbb{R}^n.\tag{3.1}$$

The following theorem then holds:

THEOREM 3.1: *Under assumption* (H), *let* $\Omega\subset\mathbb{R}^n$ *be an open bounded set with real analytic boundary. Then,* $\mathrm{Cut}_A(\Omega)$ *admits an analytic stratification.*

Proof: It suffices to show that the assumptions (H1), (H2) and (H3) of Theorem 2.2 are satisfied and then to apply it to the present situation.

Assumption (H1) is obviously true.

Let us consider (H2)(a). By Proposition 2.1, verifying (H2)(a) is reduced to verifying the existence of the solution continuous up to the boundary for the boundary value problem (2.5) and this is a well-known fact of the classical theory of elliptic boundary value problems (see e.g. [8]).

In the next section we provide a proof that Assumption (H2)(b) holds.

3.1. ON THE ANALYTICITY OF THE FUNCTION u. In this section we consider Assumption (H2)(b) concerning the analyticity of u. By Proposition 2.1 it suffices to show that the solution of (2.5) f satisfies Condition (H2)(b). As in the proof of Proposition 2.3, for $\delta \in]\delta_*, +\infty[$, we consider the solution of

$$\begin{cases} Lv = \delta v & \text{in } \Omega, \\ v = 1 & \text{on } \partial\Omega. \end{cases} \tag{3.1.1}$$

Hence, (2.2) can be rephrased as follows. For every $a > \delta_*$ and for every $x_0 \in \Omega$ there exist a neighborhood of x_0, $V \subset \mathbb{R}^n$, and $C > 0$ such that, for every $(x, \delta) \in V \times]\delta_*, a[$,

$$|\partial_x^\alpha v(x, \delta)| \le C^{|\alpha|+1} |\alpha|! \tag{3.1.2}$$

for every $\alpha \in \mathbb{N}^n$. For this purpose, it suffices to observe that the well-known proof of the analytic hypo-ellipticity for second order elliptic operators with analytic coefficients is mainly based on two ingredients:

- the elliptic estimate

$$\|\varphi\|_{H^2} \le C[\|L\varphi\|_{L^2} + \|\varphi\|_{L^2}]$$

  where φ is a test function supported on a fixed compact set,
- the estimates of the commutators of L with the derivatives of arbitrary order of the solution (using suitable localizing functions).

Using these facts one can show that if f is a solution of $Lf = g$ with g real analytic, for any x_0 there exist $r, C_1, C_2 > 0$ such that

$$\sum_{|\alpha|=k} \max_{B_r(x_0)} |\partial_x^\alpha f| \le C_1 C_2^k k!, \quad k = 0, 1, 2, \ldots.$$

We point out that the constants depend on the L^∞ norms of the derivatives (possibly of order 0) of the coefficients of the operator L.

Furthermore, the analytic hypo-ellipticity of the operator L implies the analytic hypo-ellipticity of $L - \delta$ for $\delta \in]\delta_*, a[$ and (3.1.2) follows.

It remains to show that Assumption (H3) holds.

3.2. L^∞ AND LIPSCHITZ ESTIMATES OF THE SOLUTION. We want to show that there exists $C > 0$ such that

$$\sup_{x \in \overline{\Omega}} |u(x, \varepsilon)| \le C \quad \forall \varepsilon \in]0, \varepsilon_*[. \tag{3.2.1}$$

As usual, the above bound follows by comparison with a suitable auxiliary function. For this purpose, we define

$$v_a(x) := e^{\langle x, \xi \rangle + a} \quad (x \in \Omega)$$

where ξ is a non-zero vector and a is a positive constant yet to be determined. We claim that there exists $a > 0$ (independent of ε) such that, for every $\varepsilon \in]0, \varepsilon_*[$,

$$u(x, \varepsilon) \leq v_a(x) \quad \forall x \in \Omega. \tag{3.2.2}$$

We observe that the bound (3.2.1) follows from the above claim and the fact that u is non-negative. Now, $u(x, \varepsilon) - v_a(x)$ is a continuous function defined on a compact set. Hence, if its maximum is attained at a boundary point, then (3.2.2) follows from the fact that u vanishes on $\partial\Omega$ and from the positivity of v_a. On the other hand, let us assume that the point x_0 where the maximum of the function $u - v_a$ is attained belongs to Ω. We want to show that we can choose $a > 0$ (independent of ε) such that contradiction ensues. We have

$$\nabla u(x_0, \varepsilon) = \nabla v_a(x_0) \quad \text{and} \quad \nabla^2 u(x_0, \varepsilon) \leq \nabla^2 v_a(x_0).$$

Hence,

$$\begin{aligned} 1 &= -\varepsilon\ Lu(x_0) + \langle A(x_0)\nabla u, \nabla u\rangle \\ &\geq -\varepsilon L v_a(x_0) + \langle A(x_0)\nabla v_a, \nabla v_a\rangle \\ &= \langle A(x_0)\xi, \xi\rangle[-\varepsilon v_a(x_0) + v_a(x_0)^2]. \end{aligned}$$

Now it is clear that, since L is elliptic by Assumption (3.1), it is possible to choose a large enough (independent of ε) to make the last term of the above inequality greater than 1 and then (3.2.2) follows.

3.3. (2.3) OF (H3) HOLDS WITH $\alpha = 1$. Our purpose is to show that there exists a positive constant Λ_0, independent of ε, such that, for every $\Lambda \geq \Lambda_0$,

$$u(x, \varepsilon) - u(y, \varepsilon) \leq \Lambda|x - y|, \tag{3.3.1}$$

for every $x, y \in \overline{\Omega}$. Here $u = u(x, \varepsilon)$ is a solution of the boundary value problem (2.9).

In order to do this, we first remark that if $x \in \partial\Omega$ then $u(x, \varepsilon) = 0$ and, since $u(\cdot, \varepsilon) \geq 0$ on $\overline{\Omega}$, the inequality (3.3.1) holds trivially. Hence we may assume that $(x, y) \in \Omega \times \overline{\Omega}$, in (3.3.1).

Let r denote a positive number, whose size will be determined later. Set

$$U_r = \{(x, y) \in \Omega \times \overline{\Omega} | |x - y| < r\}.$$

Let us consider first $(x, y) \in (\Omega \times \overline{\Omega}) \setminus U_r$. Then we easily see that

$$u(x, \varepsilon) - u(y, \varepsilon) \leq \frac{C}{r}|x - y| \leq \Lambda_0|x - y|,$$

with an obvious choice of $\Lambda_0 > 0$.

Next let us consider ∂U_r. We have

$$\begin{aligned}\partial U_r &= \{(x,y) \in \Omega \times \overline{\Omega} | |x-y| = r\} \cup \{(x,y) \in \Omega \times \partial\Omega | |x-y| < r\} \\ &= A_1 \cup A_2.\end{aligned}$$

If $(x,y) \in A_1$ we can argue as above and the conclusion easily follows. Assume $(x,y) \in A_2$. In this case the conclusion is implied by the estimate

$$u(x,\varepsilon) \leq \Lambda_0 d_{\partial\Omega}(x),$$

where $d_{\partial\Omega}$ denotes the distance from $\partial\Omega$ and $x \in \Omega_r = \{x \in \Omega | d_{\partial\Omega}(x) < r\}$. We emphasize that $d_{\partial\Omega}$ is a smooth function in Ω_r, if $\overline{\Omega}$ is compact and r is small enough. Moreover, we can also suppose (using, for instance, Lemma 14.17 of [8]) that there exists $\mu > 0$ such that for every $x \in \Omega_r$ and for every eigenvalue λ of $\nabla^2\delta(x)$ we have that $\lambda \leq \mu$ (i.e. the principal curvatures of $\partial\Omega$ are uniformly bounded).

Let $w(x) = \Lambda_0 d_{\partial\Omega}(x)$. Then it is easy to see that

$$\begin{cases} -\varepsilon \operatorname{tr}[A(x)\nabla^2 w] + \langle A(x)\nabla w, \nabla w\rangle > 1 & \text{in } \Omega_r, \\ w(x) \geq u(x,\varepsilon) & \text{in } \partial\Omega_r, \end{cases}$$

if $\Lambda_0 > 0$ is conveniently chosen.

This fact implies that $u(\cdot,\varepsilon) \leq w$ in Ω_r. If this were not true, then there would exist at least a point $z \in \Omega_r$ such that $u(z,\varepsilon) > w(z)$. $z \notin \partial\Omega_r$, since $u \leq w$ there. Hence $z \in \Omega_r$. This implies that the function $u(\cdot,\varepsilon) - w$ has a maximum in the interior. Let us denote by $x_M \in \Omega_r$ a point where this maximum is attained. As a consequence $\nabla u(x_M,\varepsilon) = \nabla w(x_M)$ and $\nabla^2 u(x_M,\varepsilon) \leq \nabla^2 w(x_M)$, so that $\operatorname{tr}(A(x)\nabla^2 u) \leq \operatorname{tr}(A(x)\nabla^2 w)$. But

$$\begin{aligned}-\varepsilon \operatorname{tr}(A(x_M)\nabla^2 u(x_M,\varepsilon)) &= 1 - \langle A(x_M)\nabla u(x_M,\varepsilon), \nabla u(x_M,\varepsilon)\rangle \\ &= 1 - \langle A(x_M)\nabla w(x_M), \nabla w(x_M)\rangle \\ &< -\varepsilon \operatorname{tr}(A(x_M)\nabla^2 w(x_M)),\end{aligned}$$

which yields a contradiction. Hence $u(\cdot,\varepsilon) \leq w$ in Ω_r, which proves the assertion in ∂U_r.

The next step is to prove the assertion in $\{(x,y) \in \Omega \times \Omega | |x-y| < r\}$. We use a standard computation in the theory of the viscosity solutions. Set

$$v_\varepsilon(x) := 1 - e^{-u(x,\varepsilon)}, \quad x \in \overline{\Omega}. \tag{3.3.2}$$

We observe that, by (3.2.1), the above function v_ε is uniformly bounded w.r.t. ε. Now, it is easy to see that if v_ε is uniformly Lipschitz w.r.t. ε, then u inherits a uniform Lipschitz bound from v_ε. Indeed, for every $x, y, \in \overline{\Omega}$,

$$|u(x,\varepsilon) - u(y,\varepsilon)| \le e^C |v_\varepsilon(x) - v_\varepsilon(y)| \quad \forall \varepsilon \in]0, \varepsilon_*], \tag{3.3.3}$$

where C is the constant given in formula (3.2.1). It is easy to see that since u solves equation (2.9) then v_ε is the solution of the following Dirichlet problem:

$$\begin{cases} v(x) - \varepsilon Lv(x) + \frac{1-\varepsilon}{1-v(x)} \langle A(x)\nabla v(x), \nabla v(x)\rangle = 1, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.3.4}$$

We remark that $1 - v_\varepsilon$ is bounded away from 0 uniformly w.r.t. ε, by (3.2.1). The following part of the proof has been inspired by Ishii and Lions' proof of a comparison theorem for solutions of degenerate elliptic equations in [12].

We want to show that there exists a $\Lambda_0 > 0$ such that, for every $\Lambda \ge \Lambda_0$ and every $\varepsilon \in]0, \varepsilon_*]$ ($\varepsilon_* < 1$),

$$v_\varepsilon(x) - v_\varepsilon(y) \le \Lambda |x - y| \quad \text{for every } x, y \in \Omega. \tag{3.3.5}$$

Set

$$p_{\varepsilon,L}(x,y) := v_\varepsilon(x) - v_\varepsilon(y) - \Lambda\psi(x,y), \quad (x,y) \in \Omega \times \Omega, \tag{3.3.6}$$

where Λ is the Lipschitz constant yet to be determined and

$$\psi(x,y) := |x - y|. \tag{3.3.7}$$

We argue by contradiction. Assume that for every $\lambda_0 > 0$ there are $\lambda \ge \lambda_0$ and a point $(\bar{x}, \bar{y}) \in \Omega \times \Omega$, $|\bar{x} - \bar{y}| < r$, such that

$$p_{\varepsilon,\lambda}(\bar{x}, \bar{y}) > 0.$$

We emphasize that the point $(\bar{x}, \bar{y})$ also depends on λ, even though we do not explicitly write it. Then $p_{\varepsilon,\lambda}(x,y)$ has a maximum in $\Omega \times \Omega$. It is evident that this maximum is not reached on the diagonal, since $p_{\varepsilon,\lambda}(x,x) = 0$.

Let us consider, for positive λ, the point $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ where the maximum of $p_{\varepsilon,\lambda}$ is attained. As we said above, $x_\varepsilon \ne y_\varepsilon$ and both x_ε and y_ε depend on λ.

Hence, we have that, at $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$,

$$\nabla p_{\varepsilon,\lambda} = 0 \quad \text{and} \quad \nabla^2 p_{\varepsilon,\lambda} \le 0.$$

The first equation yields

$$\nabla v_\varepsilon(x_\varepsilon) = \nabla v_\varepsilon(y_\varepsilon) = \lambda \nabla_x \psi(x_\varepsilon, y_\varepsilon) = -\lambda \nabla_y \psi(x_\varepsilon, y_\varepsilon) = \lambda \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|}, \tag{3.3.8}$$

while the inequality can be rewritten as

$$\begin{bmatrix} \nabla^2 v_\varepsilon(x_\varepsilon) & 0 \\ 0 & -\nabla^2 v_\varepsilon(y_\varepsilon) \end{bmatrix} \leq \begin{bmatrix} P & -P \\ -P & P \end{bmatrix}, \tag{3.3.9}$$

where 0 denotes the $n \times n$ null matrix and

$$P = \lambda \nabla^2_{xy} \psi(x_\varepsilon, y_\varepsilon) = \frac{\lambda}{|x_\varepsilon - y_\varepsilon|}\Big(I - \frac{(x_\varepsilon - y_\varepsilon) \otimes (x_\varepsilon - y_\varepsilon)}{|x_\varepsilon - y_\varepsilon|^2}\Big).$$

We recall that, if B and U are non-negative (symmetric) matrices, then $\operatorname{tr} BU \geq 0$. Hence, taking

$$B := \begin{bmatrix} A(x_\varepsilon) & \sqrt{A(x_\varepsilon)}\sqrt{A(y_\varepsilon)} \\ \sqrt{A(y_\varepsilon)}\sqrt{A(x_\varepsilon)} & A(y_\varepsilon) \end{bmatrix}$$

and using the inequality (3.3.9), we deduce that

$$Lv_\varepsilon(x_\varepsilon) - Lv_\varepsilon(y_\varepsilon) \leq (\sqrt{A(x_\varepsilon)} - \sqrt{A(y_\varepsilon)})^2 \cdot P \tag{3.3.10}$$

where $A \cdot B := \operatorname{tr} AB$. We recall the well-known fact that the analytic regularity (C^2 is enough) of the components of $A(\cdot)$ yields that $x \mapsto \sqrt{A(x)}$ is Lipschitz continuous. Hence, we find that

$$(\sqrt{A(x_\varepsilon)} - \sqrt{A(y_\varepsilon)})^2 \cdot P \leq C\lambda |x_\varepsilon - y_\varepsilon|, \tag{3.3.11}$$

for some constant C independent of ε. Now, using equations (3.3.4) and (3.3.10) we obtain that

$$\begin{aligned} v_\varepsilon(x_\varepsilon) - v_\varepsilon(y_\varepsilon) &+ \frac{(1-\varepsilon)\lambda^2}{1 - v_\varepsilon(x_\varepsilon)} \langle A(x_\varepsilon) \nabla_x \psi(x_\varepsilon, y_\varepsilon), \nabla_x \psi(x_\varepsilon, y_\varepsilon)\rangle \\ &\leq \frac{(1-\varepsilon)\lambda^2}{1 - v_\varepsilon(y_\varepsilon)} \langle A(y_\varepsilon) \nabla_x \psi(x_\varepsilon, y_\varepsilon), \nabla_x \psi(x_\varepsilon, y_\varepsilon)\rangle + \varepsilon(\sqrt{A(x_\varepsilon)} - \sqrt{A(y_\varepsilon)})^2 \cdot P. \end{aligned}$$

Subtracting the term

$$\frac{(1-\varepsilon)\lambda^2}{1 - v_\varepsilon(y_\varepsilon)} \langle A(x_\varepsilon) \nabla_x \psi(x_\varepsilon, y_\varepsilon), \nabla_x \psi(x_\varepsilon, y_\varepsilon)\rangle$$

from both sides of the above inequality we obtain that

(3.3.12) $(v_\varepsilon(x_\varepsilon) - v_\varepsilon(y_\varepsilon))$

$$\times\Big(1+\frac{(1-\varepsilon)\lambda^2}{(1-v_\varepsilon(x_\varepsilon))(1-v_\varepsilon(y_\varepsilon))}\langle A(x_\varepsilon)\nabla_x\psi(x_\varepsilon,y_\varepsilon),\nabla_x\psi(x_\varepsilon,y_\varepsilon)\rangle\Big)$$
$$\le\frac{(1-\varepsilon)\lambda^2}{1-v_\varepsilon(y_\varepsilon)}\langle[A(y_\varepsilon)-A(x_\varepsilon)]\nabla_x\psi(x_\varepsilon,y_\varepsilon),\nabla_x\psi(x_\varepsilon,y_\varepsilon)\rangle$$
$$+\varepsilon(\sqrt{A(x_\varepsilon)}-\sqrt{A(y_\varepsilon)})^2\cdot P,$$

i.e. (using (3.3.11))

$$(v_\varepsilon(x_\varepsilon)-v_\varepsilon(y_\varepsilon))$$
$$\times\Big(1+\frac{(1-\varepsilon)\lambda^2}{(1-v_\varepsilon(x_\varepsilon))(1-v_\varepsilon(y_\varepsilon)|x_\varepsilon-y_\varepsilon|^2)}\langle A(x_\varepsilon)(x_\varepsilon-y_\varepsilon),(x_\varepsilon-y_\varepsilon)\rangle\Big)$$
$$\le\frac{(1-\varepsilon)\lambda^2}{(1-v_\varepsilon(y_\varepsilon))|x_\varepsilon-y_\varepsilon|^2}\langle[A(y_\varepsilon)-A(x_\varepsilon)](x_\varepsilon-y_\varepsilon),(x_\varepsilon-y_\varepsilon)\rangle$$
$$+\varepsilon C\lambda|x_\varepsilon-y_\varepsilon|.$$

Hence, using (3.1) and the fact that $1-v_\varepsilon$ is uniformly bounded and away from 0 we deduce that

$$v_\varepsilon(x_\varepsilon)-v_\varepsilon(y_\varepsilon)\le C|x_\varepsilon-y_\varepsilon|$$

for some $C>0$ independent of ε. Hence, choosing $\Lambda_0=C$ we obtain a contradiction.

3.4. (2.4) OF (H3) HOLDS WITH $\beta=1/2$. We show that the second part of the assumption (H3) holds with $\beta=1/2$, i.e. there exists $c>0$ such that

$$\sup_{x\in\overline{\Omega}}|u(x,\varepsilon)-u(x,\nu)|\le c\varepsilon^{1/2},\quad 0<\nu\le\varepsilon<1. \tag{3.4.1}$$

We denote by v_ε the solution of the problem (3.3.4).

By formula (3.3.3), it suffices to find an upper bound for $v_\varepsilon-v_\nu$. We define

$$\Phi_\delta(x,y):=v_\varepsilon(x)-v_\nu(y)-|x-y|^2/2\delta,\quad (x,y)\in\overline{\Omega}\times\overline{\Omega},$$

where δ is a positive constant yet to be determined. Since Φ_δ is a continuous function on a compact set there exists $(x_0,y_0)\in\overline{\Omega}\times\overline{\Omega}$ such that

$$\max_{\overline{\Omega}\times\overline{\Omega}}\Phi_\delta=\Phi_\delta(x_0,y_0).$$

Moreover, we have that

$$v_\varepsilon(y_0)-v_\nu(y_0)\le\Phi_\delta(x_0,y_0)=v_\varepsilon(x_0)-v_\nu(y_0)-|x_0-y_0|^2/2\delta$$

and, using the Lipschitz continuity of v, we deduce that

$$|x_0-y_0|^2/2\delta\le C|x_0-y_0|$$

for some $C > 0$, i.e.

$$|x_0 - y_0| \le 2\delta C. \tag{3.4.2}$$

Now, we observe that if $x_0 \in \partial\Omega$ then

$$v_\varepsilon(x) - v_\nu(x) \le \Phi_\delta(x_0, y_0) \le 0 \quad \forall x \in \overline{\Omega}. \tag{3.4.3}$$

Moreover, if $x_0 \in \Omega$ and $y_0 \in \partial\Omega$ then

$$v_\varepsilon(y_0) - v_\nu(y_0) \le \Phi_\delta(x_0, y_0) = v_\varepsilon(x_0) - v_\nu(y_0) - |x_0 - y_0|^2/2\delta,$$

hence using once more the Lipschitz continuity of v and (3.4.2) we obtain that

$$\begin{aligned} v_\varepsilon(x) - v_\nu(x) &\le \Phi_\delta(x_0, y_0) \le v_\varepsilon(x_0) - v_\nu(y_0) \\ &\le C|x_0 - y_0| \le 2\delta C^2 \quad \forall x \in \overline{\Omega}. \end{aligned} \tag{3.4.4}$$

We are left with the case when $(x_0, y_0) \in \Omega \times \Omega$. Using the fact that (x_0, y_0) is an "interior" maximum point for the function

$$(x, y) \mapsto v_\varepsilon(x) - v_\nu(y) - |x - y|^2/2\delta$$

we obtain that

$$\nabla v_\varepsilon(x_0) = \nabla v_\nu(y_0) = \delta(x_0 - y_0)$$

and

$$-\nabla^2 v_\nu(y_0), \nabla^2 v_\varepsilon(x_0) \le \delta^{-1} I.$$

Hence,

$$v_\varepsilon(x_0) - \frac{\varepsilon}{\delta}\,\mathrm{tr}[A(x_0)] + \frac{1-\varepsilon}{\delta^2(1 - v_\varepsilon(x_0))}\langle A(x_0)(x_0 - y_0), (x_0 - y_0) \le 1 \tag{3.4.5}$$

and

$$v_\nu(y_0) + \frac{\nu}{\delta}\,\mathrm{tr}[A(y_0)] + \frac{1-\nu}{\delta^2(1 - v_\nu(y_0))}\langle A(y_0)(x_0 - y_0), (x_0 - y_0)\rangle \ge 1. \tag{3.4.6}$$

Hence, from (3.4.5) and (3.4.6) we obtain that

$$\begin{aligned} v_\varepsilon(x_0) - v_\nu(y_0) \le &\frac{\varepsilon + \nu}{\delta} \max\{\mathrm{tr}\, A(x_0), \mathrm{tr}\, A(y_0)\} \\ &+ \frac{\varepsilon - \nu}{\delta^2(1 - v_\nu(y_0))}\langle A(y_0)(x_0 - y_0), (x_0 - y_0)\rangle \\ &+ \frac{(1-\varepsilon)(v_\nu(y_0) - v_\varepsilon(x_0))}{\delta^2(1 - v_\nu(y_0))(1 - v_\varepsilon(x_0))}\langle A(y_0)(x_0 - y_0)\rangle, (x_0 - y_0)\rangle \\ &+ \frac{1-\varepsilon}{\delta^2(1 - v_\varepsilon(x_0))}\langle [A(y_0) - A(x_0)](x_0 - y_0).(x_0 - y_0)\rangle. \end{aligned}$$

Thus

$$\begin{aligned}&(v_\varepsilon(x_0)-v_\nu(y_0))\\ &\times\Big(1+\frac{(1-\varepsilon)}{\delta^2(1-v_\nu(y_0))(1-v_\varepsilon(x_0))}\langle A(y_0)(x_0-y_0),(x_0-y_0)\rangle\Big)\\ \le&\frac{\varepsilon+\nu}{\delta}\max\{\operatorname{tr}A(x_0),\operatorname{tr}A(y_0)\}\\ &+\frac{\varepsilon-\nu}{\delta^2(1-v_\nu(y_0))}\langle A(y_0)(x_0-y_0),(x_0-y_0)\rangle\\ &+\frac{1-\varepsilon}{\delta^2(1-v_\varepsilon(x_0))}\langle[A(y_0)-A(x_0)](x_0-y_0),(x_0-y_0)\rangle.\end{aligned}$$

Since the only interesting case in the above estimate is when $v_\varepsilon(x_0)-v_\nu(y_0)\ge 0$, we obtain

$$\begin{aligned}v_\varepsilon(x_0)-v_\nu(y_0)\le&\frac{\varepsilon+\nu}{\delta}\max\{\operatorname{tr}A(x_0),\operatorname{tr}A(y_0)\}\\ &+\frac{\varepsilon-\nu}{\delta^2(1-v_\nu(y_0))}\langle A(y_0)(x_0-y_0),(x_0-y_0)\rangle\\ &+\frac{1-\varepsilon}{\delta^2(1-v_\varepsilon(x_0))}\langle[A(y_0)-A(x_0)](x_0-y_0),(x_0-y_0)\rangle.\end{aligned}$$

Then, taking $\delta=\varepsilon^{1/2}$ we conclude that

$$v_\varepsilon(x_0)-v_\nu(y_0)\le C\varepsilon^{1/2}$$

for some $C>0$. The above inequality, (3.4.3) and (3.4.4) yield that

$$v_\varepsilon(x)-v_\nu(x)\le C\varepsilon^{1/2}\quad\forall x\in\overline{\Omega}.$$

Using the same argument and interchanging the roles of v_ε and v_ν, we conclude that

$$\|v_\varepsilon-v_\nu\|_\infty\le C\varepsilon^{1/2}.$$

This completes the proof of Theorem 3.1. ∎

4. A degenerate case: The Grušin operator

In this section our purpose is to apply the basic theorem to an operator of Grušin type. We use the following notation

$$x=(x',x^n),\quad x'=(x^1,\dots,x^{n-1}),$$

and for the sake of simplicity we denote by $|\cdot|$ both the standard Euclidean norm in $\mathbb{R}^n$ and the absolute value of scalars in $\mathbb{R}$; moreover we define, for $k \in \mathbb{N}$,

$$[x']_k^{2k} := \sum_{j=1}^{n-1} (x^j)^{2k}.$$

Let us consider the matrix

$$A(x', x^n) = \begin{bmatrix} I & 0 \\ 0 & [x']_k^{2k} \end{bmatrix},$$

k being the above integer, where I is the $(n-1) \times (n-1)$ identity matrix. Then, the eikonal equation is

$$\begin{cases} \sum_{j=1}^{n-1} (\partial_{x^j} d(x))^2 + [x']_k^{2k} (\partial_{x^n} d(x))^2 = 1, & x \in \Omega, \\ d(x) = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

Set

$$L = \sum_{j=1}^{n-1} (\partial_{x^j}^2 + (x^j)^{2k} \partial_{x^n}^2) = \Delta_{x'} + [x']_k^{2k} \partial_{x^n}^2. \tag{4.2}$$

We recall that the boundary $\partial\Omega$ is non-characteristic (for the operator L) if and only if, for every $x \in \partial\Omega$,

$$\sum_{j=1}^{n-1} \nu^j(x)^2 + [x']_k^{2k} \nu^n(x)^2 \neq 0,$$

where $\nu(x) = (\nu^1(x), \ldots, \nu^n(x))$ is a unit normal vector to $\partial\Omega$ at x. The following result holds.

THEOREM 4.1: *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with real analytic non-characteristic boundary. Then, $\mathrm{Cut}_A(\Omega)$ admits an analytic stratification.*

Proof: The assumption (H1) is trivially satisfied. (H2)(a) holds. Indeed, the existence of a classical solution (continuous up to the boundary) to the problem (2.5) can be proved arguing as Bony in [5] and using Proposition 2.1.

In the next section we provide a proof that Assumption (H2)(b) holds.

4.1. THE ANALYTIC HYPO-ELLIPTICITY OF THE OPERATOR L. In this section we prove that the operator (4.2) is analytic hypo-elliptic. We have

THEOREM 4.1.1: *Let*

$$L(x,D) = |D'|^2 + [x']_k^{2k} D_n^2, \tag{4.1.1}$$

where $D_j = (-1)^{-1/2}\partial/\partial x^j$ *and* $D' = (D_1, \ldots, D_{n-1})$, *so that* $|D'|^2 = -\Delta_{x'}$. *Let us consider the equation*

$$L(x,D)u = f, \tag{4.1.2}$$

where $f \in C^\omega(U)$, U *being a neighborhood of the origin in* $\mathbb{R}^n$. *Then* u *is real analytic in* U.

Proof: The proof is basically an estimate of the derivatives of u in a neighborhood of the origin.

Let us denote by φ a cut-off function identically equal to one in a neighborhood of the origin in $\mathbb{R}^n$. Due to the special form of our coordinates and the fact that the characteristic manifold is symplectic, we may assume that φ is independent of the variable x': in fact we may always take φ as a product of n such cut-off functions each depending on a single coordinate, x^j, and every x^j-derivative landing on $\varphi(x^j)$, $j = 1, \ldots, n-1$, would leave a cut-off supported in a region where x^j is bounded away from zero, hence in a region where the operator is (uniformly, microlocally) elliptic.

Thus we take $\varphi(x) = \varphi(x^n)$. Here φ is assumed to be a function of Ehrenpreis–Hörmander type (see e.g. [7], [11]), i.e. denoting by U our neighborhood of the origin. Then φ has the following property: for any $\tilde{U}$ compactly contained in U, and for any fixed $r \in \mathbb{N}$, we choose $\varphi = \varphi_r \in C_0^\infty(U)$, $\varphi \equiv 1$ on $\tilde{U}$ and such that, with a **universal** constant (i.e. depending only on the dimension of the Euclidean space in which we work) C_0,

$$|\varphi^{(k)}(x)| \leq \left(\frac{C_0}{\operatorname{dist}(\tilde{U}, U^c)}\right)^{k+1} r^k \quad \text{for } k \leq 3r.$$

Let us denote

$$X_j = D_{x^j}, \quad Y_j = (x^j)^k \partial_{x^n}, \quad j = 1, \ldots, n-1.$$

Then

$$L(x,D) = \sum_{j=1}^{n-1} (X_j^2 + Y_j^2). \tag{4.1.3}$$

It is a well-known fact that the operator L in (4.1.3) is C^∞-hypo-elliptic and satisfies an a priori estimate of the form

$$\sum_{j=1}^{n-1}(\|X_j u\|^2 + \|Y_j u\|^2) + \|u\|^2_{1/(k+1)} \le C(|\langle Lu, u\rangle| + \|u\|^2), \tag{4.1.4}$$

where u is a rapidly decreasing smooth function, $\|\cdot\|_s$ denotes the usual Sobolev norm of order s and $\|\cdot\| = \|\cdot\|_0$ is the L^2 norm.

We want to obtain a bound for an expression of the form

$$\|X_j\varphi(x^n)D_n^r u\| \quad \text{or} \quad \|Y_j\varphi(x^n)D_n^r u\| \tag{4.1.5}$$

where, since we are in a microlocal neighborhood of the point $(0, e_n)$, D_n is an elliptic operator. It is well known that obtaining a bound for the quantities in (4.1.5) of the type $\|X_j\varphi(x^n)D_n^r u\| \le C^{r+1}r!$ allows us to deduce that L is (micro)analytic hypo-elliptic, so that the solution u of (4.1.2) is real analytic.

Let us examine first the quantity

$$\|X_j\varphi(x^n)D_n^r u\|.$$

Using the a priori inequality (4.1.4) we have

$$\|X_j\varphi(x^n)D_n^r u\|^2 \le |\langle L\varphi(x^n)D_n^r u, \varphi(x^n)D_n^r u\rangle| + \|\varphi(x^n)D_n^r u\|^2.$$

Let us consider the scalar product on the r.h.s. of the above formula. We can write

$$\begin{aligned}|\langle L\varphi(x^n)D_n^r u, \varphi(x^n)D_n^r u\rangle| \le & |\langle \varphi(x^n D_n^r Lu, \varphi(x^n)D_n^r u|\rangle| \\ & + |\langle [L, \varphi(x^n)D_n^r]u, \varphi(x^n)D_n^r u\rangle|.\end{aligned}$$

The first term above involves Lu, which by assumption is real analytic and has thus the right estimates modulo the use of the Cauchy–Schwartz inequality and reabsorbing on the left the r.h.s. term in the scalar product. We are then left with the commutator term. We have

$$[L, \varphi D_n^r] = \sum_{j=1}^{n-1}(Y_j[Y_j, \varphi D_n^r] + [Y_j, \varphi D_n^r]Y_j),$$

because the fields X_j commute with φD_n^r. Moreover,

$$[Y_j, \varphi D_n^r] = (x^j)^k \varphi' D_n^r.$$

Hence we have

$$|\langle [L,\varphi(x^n)D_n^r]u, \varphi(x^n)D_n^r u\rangle|$$
$$\leq \sum_{j=1}^{n-1}(|\langle Y_j(x^j)^k\varphi' D_n^r u, \varphi D_n^r u\rangle| + |\langle (x^j)^k\varphi' D_n^r Y_j u, \varphi D_n^r u\rangle|).$$

Let us consider a generic term in the first sum above. Since Y_j is self-adjoint we may bring Y_j to the right and, using the Cauchy–Schwartz inequality, we are reduced to estimating $\|(x^j)^k\varphi' D_n^r u\|$. Now

$$\varphi' D_n^r u = \sum_{\ell=1}^{r}(-1)^{\ell-1}D_n\varphi^{(\ell)}D_n^{r-\ell}u + (-1)^r\varphi^{(r+1)}u,$$

where $\varphi^{(\ell)} = D_n^\ell\varphi$, whence

$$\begin{aligned}\|(x^j)^k\varphi' D_n^r u\| &\leq \sum_{\ell=1}^{r}\|(x^j)^k D_n\varphi^{(\ell)}D_n^{r-\ell}u\| + \|(x^j)^k\varphi^{(r+1)}u\| \\ &= \sum_{\ell=1}^{r}\|Y_j\varphi^{(\ell)}D_n^{r-\ell}u\| + \|(x^j)^k\varphi^{(r+1)}u\|.\end{aligned} \tag{4.1.6}$$

Summarizing, we started off estimating $\|X\varphi D_n^r u\|$ and we wind up with $\|Y\varphi^{(\ell)}D_n^{r-\ell}u\|$ where $\ell \geq 1$. Iterating, we see easily that we obtain a number of terms similar to the last in (4.1.6). Furthermore, the number of those terms is bounded by C^r, where C is a positive constant. This yields analytic estimates, due to the definition of φ.

Let us now consider the term $|\langle (x^j)^k\varphi' D_n^r Y_j u, \varphi D_n^r u\rangle|$. The power $(x_j)^k$ can harmlessly go to the right of the scalar product. Thus if we pull an x_n-derivative in front of the right factor of the scalar product, we obtain the same estimate as above. The only difficulty in pulling a derivative in front is the commutation with φ'. This is achieved in a way analogous to the above.

Estimating terms of the form $\|Y_j\varphi D_n^r u\|$ is completely analogous. Hence we obtain

$$\sum_{j=1}^{n-1}(\|X_j\varphi D_n^r u\| + \|Y_j\varphi D_n^r u\|) \leq C^{r+1}r!, \tag{4.1.7}$$

which ends the proof of Theorem 4.1.1. ∎

Remark 4.1.1: Condition (H2)(b) is a direct consequence of the above proof.

In what follows we prove that Assumption (H3) holds for L.

We would like to explicitly point out that the proof of the Lipschitz estimates are carried out along the same lines of the corresponding proofs in the elliptic case. Since, however, the degeneracy of the principal symbol plays a crucial role and forces us to modify the argument in this instance, we deemed it useful to include the whole argument.

4.2. THE L^∞ ESTIMATE. The L^∞ estimate for u follows arguing as in the proof of formula (3.2.1) in Theorem 3.1, choosing ξ equal to the unit vector $(1, 0, \ldots, 0)$. Hence, also the solutions of the problem (3.3.4), v_ε, satisfy a uniform L^∞ estimate.

4.3. (2.3) OF (H3) HOLDS WITH $\alpha = 1$. We want to show that there exists a positive constant Λ_0, independent of ε, such that, for every $\Lambda \geq \Lambda_0$,

$$u(x, \varepsilon) - u(y, \varepsilon) \leq \Lambda |x - y|, \tag{4.3.1}$$

for every x, $y \in \overline{\Omega}$. Here $u = u(x, \varepsilon)$ is a solution of the boundary value problem (2.9).

In order to do this, we first remark that if $x \in \partial\Omega$ then $u(x, \varepsilon) = 0$ and, since $u(\cdot, \varepsilon) \geq 0$ on $\overline{\Omega}$, the inequality (4.3.1) holds trivially. Hence we may assume that $(x, y) \in \Omega \times \overline{\Omega}$, in (4.3.1).

Let r denote a positive number, whose size will be determined later. Set

$$U_r = \{(x, y) \in \Omega \times \overline{\Omega} | |x - y| < r\}.$$

Let us consider first $(x, y) \in (\Omega \times \overline{\Omega}) \setminus U_r$. Then we easily see that

$$u(x, \varepsilon) - u(y, \varepsilon) \leq \frac{C}{r} |x - y| \leq \Lambda_0 |x - y|,$$

with an obvious choice of $\Lambda_0 > 0$.

Next let us consider ∂U_r. We have

$$\begin{aligned} \partial U_r &= \{(x, y) \in \Omega \times \overline{\Omega} | |x - y| = r\} \cup \{(x, y) \in \Omega \times \partial\Omega | |x - y| < r\} \\ &= A_1 \cup A_2. \end{aligned}$$

If $(x, y) \in A_1$ we can argue as above and the conclusion easily follows. Assume $(x, y) \in A_2$. In this case the conclusion is implied by the estimate

$$u(x, \varepsilon) \leq \Lambda_0 d_{\partial\Omega}(x),$$

where $d_{\partial\Omega}$ denotes the distance from $\partial\Omega$ and $x \in \Omega_r = \{x \in \Omega | d_{\partial\Omega}(x) < r\}$. We emphasize that $d_{\partial\Omega}$ is a smooth function in Ω_r, if $\overline{\Omega}$ is compact and r is small enough. Moreover, we can also suppose (using, for instance, Lemma 14.17 of [8]) that there exists $\mu > 0$ such that

$$\forall x \in \Omega_r, \quad \forall \lambda \text{ eigenvalue of } \nabla^2 d_{\partial\Omega}(x), \quad \lambda \leq \mu \tag{4.3.2}$$

(i.e. the principal curvatures of $\partial\Omega$ are uniformly bounded).

We claim that there exists $c > 0$ such that, for $r > 0$ small enough,

$$\langle A(x)\nabla d_{\partial\Omega}(x), \nabla d_{\partial\Omega}(x)\rangle \geq c \quad \forall x \in \Omega_r. \tag{4.3.3}$$

We prove the above claim arguing by contradiction. Let us assume that there exists a sequence $x_j \in \Omega$ converging to a boundary point $\overline{x} \in \partial\Omega$ such that

$$\lim_{j\to\infty} \langle A(x_j)\nabla d_{\partial\Omega}(x_j), \nabla d_{\partial\Omega}(x_j)\rangle = 0. \tag{4.3.4}$$

Then, the first $n-1$ components of $\nabla d_{\partial\Omega}(x_j)$ converge to 0 (as $j \to \infty$). Moreover, using the fact that $|\nabla d_{\partial\Omega}(x_j)| = 1$, we deduce that

$$\lim_{j\to\infty} |\partial_{x^n} d_{\partial\Omega}(x_j)| = 1$$

and, using (4.3.4), we conclude that

$$\lim_{j\to\infty} x'_j = 0.$$

Thus we obtain

$$\langle A(\overline{x})\nu(\overline{x}), \nu(\overline{x})\rangle = 0,$$

where $\nu(\bar{x})$ denotes the normal to $\partial\Omega$ at $\bar{x}$. This is a contradiction.

Let $w(x) = \Lambda_0 d_{\partial\Omega}(x)$. Then

$$\begin{cases} -\varepsilon \operatorname{tr}[A(x)\nabla^2 w] + \langle A(x)\nabla w, \nabla w\rangle > 1 & \text{in } \Omega_r, \\ w(x) \geq u(x,\varepsilon) & \text{in } \partial\Omega_r, \end{cases} \tag{4.3.5}$$

if $\Lambda_0 > 0$ is conveniently chosen. Indeed, in Ω_r,

$$\begin{aligned} -\varepsilon \operatorname{tr}[A(x)\nabla^2 w] + \langle A(x)\nabla w, \nabla w\rangle &= -\varepsilon\Lambda_0 \operatorname{tr}[A(x)\nabla^2 d_{\partial\Omega}] + \Lambda_0^2\langle A(x)\nabla d_{\partial\Omega}, \nabla d_{\partial\Omega}\rangle \\ &\geq -\varepsilon\Lambda_0 c_1 + \Lambda_0^2 c \end{aligned}$$

for some $c_1 > 0$ depending on the operator L and on the bound of the boundary curvature, μ, in (4.3.2), since in the last inequality above (4.3.2) and (4.3.3) have been used.

Hence, it is clear that taking Λ_0 large enough (uniformly w.r.t. ε) we have $-\varepsilon\Lambda_0 c_1 + \Lambda_0^2 c > 1$ and (4.3.5) follows. This fact implies that $u(\cdot,\varepsilon) \leq w$ in

Ω_r. If this were not true, then there would exist at least a point $z \in \Omega_r$ such that $u(z,\varepsilon) > w(z)$. Also $z \notin \partial\Omega_r$, since $u \leq w$ there. Hence $z \in \Omega_r$. This implies that the function $u(\cdot,\varepsilon) - w$ has a maximum in the interior. Let us denote by $x_M \in \Omega_r$ a point where this maximum is attained. As a consequence $\nabla u(x_M,\varepsilon) = \nabla w(x_M)$ and $\nabla^2 u(x_M,\varepsilon) \leq \nabla^2 w(x_M)$, so that $\operatorname{tr}(A(x)\nabla^2 u) \leq \operatorname{tr}(A(x)\nabla^2 w)$. But

$$\begin{aligned} -\varepsilon \operatorname{tr}(A(x_M)\nabla^2 u(x_M,\varepsilon)) &= 1 - \langle A(x_M)\nabla u(x_M,\varepsilon), \nabla u(x_M,\varepsilon)\rangle \\ &= 1 - \langle A(x_M)\nabla w(x_M), \nabla w(x_M)\rangle \\ &< -\varepsilon \operatorname{tr}(A(x_M)\nabla^2 w(x_M)), \end{aligned}$$

which yields a contradiction. Hence $u(\cdot,\varepsilon) \leq w$ in Ω_r, which proves the assertion in ∂U_r.

The next step is to prove the assertion in $\{(x,y) \in \Omega \times \Omega \,|\, |x-y| < r\}$. Set

$$v_\varepsilon(x) := 1 - e^{-u(x,\varepsilon)}, \quad x \in \overline{\Omega}. \tag{4.3.6}$$

We observe that, by (3.2.1), the above function v_ε is uniformly bounded w.r.t. ε. Now, it is easy to see that if v_ε is uniformly Lipschitz w.r.t. ε, then u inherits a uniform Lipschitz bound from v_ε. Indeed, for every $x, y, \in \overline{\Omega}$,

$$|u(x,\varepsilon) - u(y,\varepsilon)| \leq e^C |v_\varepsilon(x) - v_\varepsilon(y)| \quad \forall \varepsilon \in \,]0, \varepsilon_*], \tag{4.3.7}$$

where C is the constant given in formula (3.2.1). It is easy to see that since u solves equation (2.9), then v_ε is the solution of the following Dirichlet problem:

$$\begin{cases} v(x) - \varepsilon L v(x) + \frac{1-\varepsilon}{1-v(x)}\langle A(x)\nabla v(x), \nabla v(x)\rangle = 1, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{4.3.8}$$

We remark that $1 - v_\varepsilon$ is bounded away from 0 uniformly w.r.t. ε, by the L^∞ estimate.

We want to show that there exists a $\Lambda_0 > 0$ such that, for every $\Lambda \geq \Lambda_0$ and every $\varepsilon \in \,]0, \varepsilon^*]$ ($\varepsilon^* < 1$),

$$v_\varepsilon(x) - v_\varepsilon(y) \leq \Lambda |x-y| \quad \text{for every } x, y \in \Omega. \tag{4.3.9}$$

Set

$$p_{\varepsilon,L}(x,y) := v_\varepsilon(x) - v_\varepsilon(y) - \Lambda\psi(x,y), \quad (x,y) \in \Omega \times \Omega, \tag{4.3.10}$$

where Λ is the Lipschitz constant yet to be determined and

$$\psi(x,y) := |x-y|. \tag{4.3.11}$$

We argue by contradiction. Assume that for every $\lambda_0 > 0$ there are $\lambda \geq \lambda_0$ and a point $(\bar{x}, \bar{y}) \in \Omega \times \Omega$, $|\bar{x} - \bar{y}| < r$, such that

$$p_{\varepsilon,\lambda}(\bar{x}, \bar{y}) > 0.$$

We emphasize that the point $(\bar{x}, \bar{y})$ also depends on λ, even though we do not explicitly write it. Then $p_{\varepsilon,\lambda}(x, y)$ has a maximum in $\Omega \times \Omega$. It is evident that this maximum is not reached on the diagonal, since $p_{\varepsilon,\lambda}(x, x) = 0$.

Let us consider, for positive λ, the point $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ where the maximum of $p_{\varepsilon,\lambda}$ is attained. As we said above $x_\varepsilon \neq y_\varepsilon$, and both x_ε and y_ε depend on λ.

Hence we have that, at $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$,

$$\nabla p_{\varepsilon,\lambda} = 0 \quad \text{and} \quad \nabla^2 p_{\varepsilon,\lambda} \leq 0.$$

The first equation yields

$$\nabla v_\varepsilon(x_\varepsilon) = \nabla v_\varepsilon(y_\varepsilon) = \lambda \nabla_x \psi(x_\varepsilon, y_\varepsilon) = -\lambda \nabla_y \psi(x_\varepsilon, y_\varepsilon) = \lambda \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|}, \tag{4.3.12}$$

while the inequality can be rewritten as

$$\begin{bmatrix} \nabla^2 v_\varepsilon(x_\varepsilon) & 0 \\ 0 & -\nabla^2 v_\varepsilon(y_\varepsilon) \end{bmatrix} \leq \begin{bmatrix} P & -P \\ -P & P \end{bmatrix}, \tag{4.3.13}$$

where 0 denotes the $n \times n$ null matrix and

$$P = \lambda \nabla^2_{xy} \psi(x_\varepsilon, y_\varepsilon) = \frac{\lambda}{|x_\varepsilon - y_\varepsilon|} \Big(I - \frac{(x_\varepsilon - y_\varepsilon) \otimes (x_\varepsilon - y_\varepsilon)}{|x_\varepsilon - y_\varepsilon|^2} \Big).$$

Using the inequality (4.3.13) we deduce

$$L v_\varepsilon(x_\varepsilon) - L v_\varepsilon(y_\varepsilon) \leq (\sqrt{A(x_\varepsilon)} - \sqrt{A(y_\varepsilon)})^2 \cdot P \tag{4.3.14}$$

and we find that

$$(\sqrt{A(x_\varepsilon)} - \sqrt{A(y_\varepsilon)})^2 \cdot P \leq C \frac{\lambda}{|x_\varepsilon - y_\varepsilon|} ([x'_\varepsilon]^k_k - [y'_\varepsilon]^k_k)^2, \tag{4.3.15}$$

for some constant C independent of ε. Now, using equations (4.3.8) and (4.3.14) we obtain that

$$\begin{aligned} & v_\varepsilon(x_\varepsilon) - v_\varepsilon(y_\varepsilon) + \frac{(1-\varepsilon)\lambda^2}{1 - v_\varepsilon(x_\varepsilon)} \langle A(x_\varepsilon) \nabla_x \psi(x_\varepsilon, y_\varepsilon), \nabla_x \psi(x_\varepsilon, y_\varepsilon) \rangle \\ & \leq \frac{(1-\varepsilon)\lambda^2}{1 - v_\varepsilon(y_\varepsilon)} \langle A(y_\varepsilon) \nabla_x \psi(x_\varepsilon, y_\varepsilon), \nabla_x \psi(x_\varepsilon, y_\varepsilon) \rangle + \varepsilon (\sqrt{A(x_\varepsilon)} - \sqrt{A(y_\varepsilon)})^2 \cdot P. \end{aligned}$$

Subtracting the term

$$\frac{(1-\varepsilon)\lambda^2}{1-v_\varepsilon(x_\varepsilon)}\langle A(y_\varepsilon)\nabla_x\psi(x_\varepsilon,y_\varepsilon),\nabla_x\psi(x_\varepsilon,y_\varepsilon)\rangle$$

from both sides of the above inequality we obtain

$$\begin{aligned}(4.3.16)\quad &(v_\varepsilon(x_\varepsilon)-v_\varepsilon(y_\varepsilon))\\ &\times\Big(1+\frac{(1-\varepsilon)\lambda^2}{(1-v_\varepsilon(x_\varepsilon))(1-v_\varepsilon(y_\varepsilon))}\langle A(y_\varepsilon)\nabla_x\psi(x_\varepsilon,y_\varepsilon),\nabla_x\psi(x_\varepsilon,y_\varepsilon)\rangle\Big)\\ &\le\frac{(1-\varepsilon)\lambda^2}{1-v_\varepsilon(x_\varepsilon)}\langle[A(y_\varepsilon)-A(x_\varepsilon)]\nabla_x\psi(x_\varepsilon,y_\varepsilon),\nabla_x\psi(x_\varepsilon,y_\varepsilon)\rangle\\ &+\varepsilon(\sqrt{A(x_\varepsilon)}-\sqrt{A(y_\varepsilon)})^2\cdot P,\end{aligned}$$

i.e. (using (4.3.15))

$$\begin{aligned}(4.3.17)\quad &(v_\varepsilon(x_\varepsilon)-v_\varepsilon(y_\varepsilon))\\ &\times\Big(1+\frac{(1-\varepsilon)\lambda^2}{(1-v_\varepsilon(x_\varepsilon))(1-v_\varepsilon(y_\varepsilon))|x_\varepsilon-y_\varepsilon|^2}\langle A(y_\varepsilon)(x_\varepsilon-y_\varepsilon),(x_\varepsilon-y_\varepsilon)\rangle\Big)\\ &\le\frac{(1-\varepsilon)\lambda^2}{(1-v_\varepsilon(x_\varepsilon))|x_\varepsilon-y_\varepsilon|^2}\langle[A(y_\varepsilon)-A(x_\varepsilon)](x_\varepsilon-y_\varepsilon),(x_\varepsilon-y_\varepsilon)\rangle\\ &+\varepsilon C\frac{\lambda}{|x_\varepsilon-y_\varepsilon|}([x'_\varepsilon]_k^k-[y'_\varepsilon]_k^k)^2.\end{aligned}$$

To estimate $v_\varepsilon(x_\varepsilon)-v_\varepsilon(y_\varepsilon)$ two cases may occur:

Case 1:

$$(4.3.18)\qquad [y'_\varepsilon]_k\ge[x'_\varepsilon]_k$$

(i.e. $A(y_\varepsilon)-A(x_\varepsilon)\ge 0$). In this case, using inequality (4.3.17), the fact that $1-v_\varepsilon$ is uniformly bounded and away from 0 and that λ is large, the estimate of $v_\varepsilon(x_\varepsilon)-v_\varepsilon(y_\varepsilon)$ reduces to estimating the behavior of the following two terms:

$$(\mathrm{I})=\frac{\langle[A(y_\varepsilon)-A(x_\varepsilon)](x_\varepsilon-y_\varepsilon),(x_\varepsilon-y_\varepsilon)\rangle}{\langle A(y_\varepsilon)(x_\varepsilon-y_\varepsilon),(x_\varepsilon-y_\varepsilon)\rangle}$$

and

$$(\mathrm{II})=\frac{\varepsilon|x_\varepsilon-y_\varepsilon|([x'_\varepsilon]_k^k-[y'_\varepsilon]_k^k)^2}{\langle A(y_\varepsilon)(x_\varepsilon-y_\varepsilon),(x_\varepsilon-y_\varepsilon)\rangle}.$$

For the sake of brevity in what follows we omit to write the subscript ε. We have

$$(4.3.19)\qquad (\mathrm{I})=\frac{([y']_k^{2k}-[x']_k^{2k})(x_n-y_n)^2}{\sum_{j=1}^{n-1}(x_j-y_j)^2+[y']_k^{2k}(x_n-y_n)^2}$$

$$\leq \frac{([y']_k^{2k} - [x']_k^{2k})(x_n - y_n)^2}{\sqrt{\sum_{j=1}^{n-1}(x_j - y_j)^2}[y']_k^k|x_n - y_n|}$$

so that

$$\begin{aligned}
(\mathrm{I}) &\leq \frac{(x_n - y_n)^2 \sum_{j=1}^{n-1}(y_j^{2k} - x_j^{2k})}{|x_n - y_n|\sqrt{\sum_{j=1}^{n-1}(x_j - y_j)^2}\sqrt{\sum_{j=1}^{n-1} y_j^{2k}}} \\
&\leq \frac{|x_n - y_n|\sqrt{\sum_{j=1}^{n-1}(y_j - x_j)^2}\sqrt{\sum_{j=1}^{n-1}(\sum_{i=0}^{2k-1} y_j^{2k-1-i}x_j^i)^2}}{\sqrt{\sum_{j=1}^{n-1}(x_j - y_j)^2}\sqrt{\sum_{j=1}^{n-1} y_j^{2k}}} \\
&\leq \frac{|x_n - y_n|\sqrt{\sum_{j=1}^{n-1}(\sum_{i=0}^{2k-1} y_j^{2k-1-i}x_j^i)^2}}{\sqrt{\sum_{j=1}^{n-1} y_j^{2k}}} \\
&\leq c_{nk}|x_n - y_n|[y']_k^{k-1},
\end{aligned}$$

for some positive c_{nk} depending only on the dimension n and the order k. Moreover,

$$(\mathrm{II}) \leq \varepsilon \frac{\sqrt{\sum_{j=1}^{n}(x_j - y_j)^2}\sqrt{\sum_{j=1}^{n-1}(x_j^k - y_j^k)^2}}{\sqrt{\sum_{j=1}^{n-1}(x_j - y_j)^2}} \leq C[y']_k^{2(k-1)}|x - y|,$$

for some C depending only on k and Ω.

The other possibility which might occur is

Case 2:

$$[y'_\varepsilon]_k < [x'_\varepsilon]_k \tag{4.3.20}$$

(i.e. $A(y_\varepsilon) - A(x_\varepsilon) \leq 0$). Using the inequality (4.3.17), we obtain that

$$\begin{aligned}
&v_\varepsilon(x_\varepsilon) - v_\varepsilon(y_\varepsilon) \\
&\leq \Big(1 + \frac{(1-\varepsilon)\lambda^2}{(1 - v_\varepsilon(x_\varepsilon))(1 - v_\varepsilon(y_\varepsilon))|x_\varepsilon - y_\varepsilon|^2}\langle A(x_\varepsilon)(x_\varepsilon - y_\varepsilon), (x_\varepsilon - y_\varepsilon)\rangle\Big)^{-1} \\
&\qquad\qquad \times \varepsilon C \frac{\lambda}{|x_\varepsilon - y_\varepsilon|}([x'_\varepsilon]_k^k - [y'_\varepsilon]_k^k)^2
\end{aligned}$$

and the estimate of $v_\varepsilon(x_\varepsilon) - v_\varepsilon(y_\varepsilon)$ reduces to

$$\begin{aligned}
\frac{\varepsilon|x - y|\sum_{j=1}^{n-1}(x_j^k - y_j^k)^2}{\sum_{j=1}^{n-1}(x_j - y_j)^2 + \sum_{j=1}^{n-1} x_j^k(x_n - y_n)^2} &\leq \frac{\varepsilon|x - y|\sum_{j=1}^{n-1}(x_j^k - y_j^k)^2}{\sum_{j=1}^{n-1}(x_j - y_j)^2} \\
&\leq c|x - y|,
\end{aligned}$$

for some c independent of ε.

Hence in both the cases, we deduce that

$$v_\varepsilon(x_\varepsilon) - v_\varepsilon(y_\varepsilon) \leq C|x_\varepsilon - y_\varepsilon|$$

for some $C > 0$ independent of ε. Thus, choosing $\Lambda_0 = C$, we obtain a contradiction.

Finally, arguing as in Section 3.3, we find that (2.4) of (H3) holds with $\beta = 1/2$. This completes the proof of Theorem 4.1. ∎

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